Entanglement in Gaussian matrix-product states

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Gaussian matrix product states are obtained as the outputs of projection operations from an ancillary space of M infinitely entangled bonds connecting neighboring sites, applied at each of N sites of an harmonic chain. Replacing the projections by associated Gaussian states, the *building blocks*, we show that the entanglement range in translationally-invariant Gaussian matrix product states depends on how entangled the building blocks are. In particular, infinite entanglement in the building blocks produces fully symmetric Gaussian states with maximum entanglement range. From their peculiar properties of entanglement sharing, a basic difference with spin chains is revealed: Gaussian matrix product states can possess unlimited, long-range entanglement even with minimum number of ancillary bonds (M=1). Finally we discuss how these states can be experimentally engineered from N copies of a three-mode building block and N two-mode finitely squeezed states.

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Introduction.— The description of many-body systems and the understanding of multiparticle entanglement are among the hardest challenges of quantum physics. The two issues are entwined: recently, the basic tools of quantum information theory have found useful applications in condensed matter physics. In particular, the formalism of *matrix product states* (MPS) [1] has led to an efficient simulation of many-body spin Hamiltonians [2] and to a deeper understanding of quantum phase transitions [3].

Beyond qubits, and discrete-variable systems in general, a growing interest is being witnessed in the theoretical and experimental applications of so-called continuous variable (CV) systems, such as ultracold atoms or light modes, to quantum information and communication processings [4]. Besides their usefulness in feasible implementations, quasi-free states of harmonic lattices, best known as Gaussian states, are endowed with structural properties that make the characterization of their entanglement amenable to an analytical analysis [5]. Within this context, the extension of the matrix product framework to Gaussian states of CV systems has been recently introduced as a possible tool to prove an entropic area law for critical bosonic systems on harmonic lattices [6], thus complementing the known results for the non-critical case [7].

In this work we adopt a novel point of view, aimed to comprehend the correlation picture of the considered many-body systems from the physical structure of the underlying MPS framework. In the case of harmonic lattices, we demonstrate that the quantum correlation length (the maximum distance between pairwise entangled sites) of translationally invariant Gaussian MPS is determined by the amount of entanglement encoded in a smaller structure, the 'building block', which is a Gaussian state isomorphic to the MPS projector at each site. This connection provides a series of necessary and sufficient conditions for bipartite entanglement of distant pair of modes in Gaussian MPS depending on the parameters of the building block, as explicitly shown for a six-mode harmonic ring. For any size of the ring we show remarkably that, when

single ancillary bonds connect neighboring sites, an infinite entanglement in the building block leads to fully symmetric (permutation-invariant) Gaussian MPS where each individual mode is equally entangled with any other, independently of the distance. As the block entropy of these states can diverge for any bipartition of the ring [9], our results unveil a basic difference with finite-dimensional MPS, whose entanglement is limited by the bond dimensionality [10] and is typically short-ranged [11]. Finally, we demonstrate how to experimentally implement the MPS construction to produce multimode Gaussian states useful for CV communication networks [12].

Gaussian matrix product states.— In a CV system consisting of N canonical bosonic modes, described by the vector $\hat{R} = \{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N, \hat{p}_1, \hat{p}_2 \ldots, \hat{p}_N\}$ of the field quadrature operators, Gaussian states (such as coherent and squeezed states) are fully characterized by the first statistical moments (arbitrarily adjustable by local unitaries: we will set them to zero) and by the $2N \times 2N$ real symmetric covariance matrix (CM) $\boldsymbol{\gamma}$ of the second moments $\gamma_{ij} = 1/2\langle\{\hat{R}_i, \hat{R}_j\}\rangle$ [5].

The Gaussian matrix product states introduced in Ref. [6] are N-mode states obtained by taking a fixed number, M, of infinitely entangled ancillary bonds (EPR pairs) shared by adjacent sites, and applying an arbitrary $2M \to 1$ Gaussian operation on each site $i = 1, \ldots, N$. Here the *cardinality M* is the CV counterpart of the dimension D of the matrices in standard MPS [1]. Such a process can be better understood by resorting to the Jamiolkowski isomorphism between (Gaussian) operations and (Gaussian) states [8]. In this framework, one starts with a chain of N Gaussian states of $2M+1$ modes: the *building blocks*. The global Gaussian state of the chain is described by a CM $\boldsymbol{\Gamma} = \bigoplus_{i=1}^{N} \boldsymbol{\gamma}^{[i]}$. As the interest in MPS lies mainly in their connections with ground states of Hamiltonians invariant under translation [6], we can focus on pure ($\text{Det}\,\boldsymbol{\gamma}^{[i]} = 1$), translationally invariant ($\boldsymbol{\gamma}^{[i]} \equiv \boldsymbol{\gamma}\,\forall i$) Gaussian MPS. Moreover, in this work we consider single-bonded MPS, i.e. with $M = 1$. However, our analysis easily general-
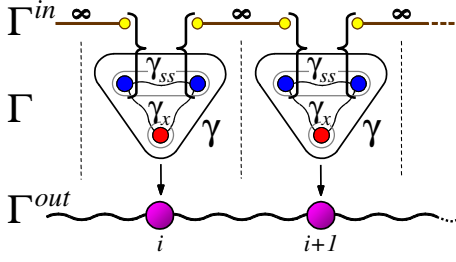
FIG. 1: (Color online) Gaussian matrix product states. $\mathbf{\Gamma}^{in}$ is the state of N EPR bonds and γ is the three-mode building block. After the EPR measurements (depicted as curly brackets), the modes γ_x collapse into a Gaussian MPS with global state $\mathbf{\Gamma}^{out}$.

izes to multiple bonds, and to mixed Gaussian states as well.

Under the considered prescriptions, the building block $\boldsymbol{\gamma}$ is a pure Gaussian state of the three modes with respective CM $\boldsymbol{\alpha}_{1,2,3}$. As we aim to construct a translationally invariant state, it is convenient to consider a $\boldsymbol{\gamma}$ whose first two modes have the same reduced CM. This yields a bisymmetric [9], pure, three-mode Gaussian building block whose CM $\boldsymbol{\gamma}$ can be written without loss of generality in standard form [13], with $(\mathrm{Det}\,\boldsymbol{\alpha}_1)^{1/2} = (\mathrm{Det}\,\boldsymbol{\alpha}_2)^{1/2} \equiv s$ and $(\mathrm{Det}\,\boldsymbol{\alpha}_3)^{1/2} \equiv x$. This choice of the building block is physically motivated by the fact that, among all pure three-mode Gaussian states, bisymmetric states maximize the genuine tripartite entanglement [13]. It is instructive to write $\boldsymbol{\gamma}$ in the block form

$$\boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_{ss} & \boldsymbol{\gamma}_{sx} \\ \boldsymbol{\gamma}_{sx}^T & \boldsymbol{\gamma}_x \end{pmatrix}, \qquad (1)$$

where $\boldsymbol{\gamma}_{ss}$ is the CM of modes 1 and 2, $\boldsymbol{\gamma}_x$ is the CM of mode 3, and the intermodal correlations are encoded in $\boldsymbol{\gamma}_{sx}$. Explicitly [13]: $\boldsymbol{\gamma}_{ss} = \begin{pmatrix} s & t_+ \\ t_+ & s \end{pmatrix} \oplus \begin{pmatrix} s & t_- \\ t_- & s \end{pmatrix}$, with $t_\pm = [x^2 - 1 \pm \sqrt{16s^4 - 8(x^2+1)s^2 + (x^2-1)^2}]/(4s)$; $\boldsymbol{\gamma}_x = \mathrm{diag}\{x, x\}$; and $\boldsymbol{\gamma}_{sx}^T = \begin{pmatrix} u_+ & u_+ & 0 & 0 \\ 0 & 0 & u_- & u_- \end{pmatrix}$, with $u_\pm = \frac{1}{4}\sqrt{\frac{x^2-1}{sx}}\left[\sqrt{(x-2s)^2 - 1} \pm \sqrt{(x+2s)^2 - 1}\right]$.

The MPS construction works as follows (see Fig. 1). The global CM $\mathbf{\Gamma} = \bigoplus_{i=1}^N \boldsymbol{\gamma}$ corresponds to the projector from the state $\mathbf{\Gamma}^{in}$ of the N ancillary EPR pairs, to the final N-mode Gaussian MPS $\mathbf{\Gamma}^{out}$. This is realized by collapsing the state $\mathbf{\Gamma}^{in}$, transposed in phase space, with the input port $\mathbf{\Gamma}_{ss} = \bigoplus_i \boldsymbol{\gamma}_{ss}$ of $\mathbf{\Gamma}$, so that the output port $\mathbf{\Gamma}_x = \bigoplus_i \boldsymbol{\gamma}_x$ turns into the desired $\mathbf{\Gamma}^{out}$. Here collapsing means that, at each site, the two two-mode states, each constituted by one mode (1 or 2) of $\boldsymbol{\gamma}_{ss}$ and one half of the EPR bond between site i and its neighbor ($i-1$ or $i+1$, respectively), undergo an "EPR measurement" i.e. are projected onto the infinitely entangled EPR state [6, 8]. An EPR pair is described by a CM

$$\boldsymbol{\sigma}_{i,j}(r) = \begin{pmatrix} \cosh(2r) & \sinh(2r) \\ \sinh(2r) & \cosh(2r) \end{pmatrix} \oplus \begin{pmatrix} \cosh(2r) & -\sinh(2r) \\ -\sinh(2r) & \cosh(2r) \end{pmatrix}, \quad (2)$$

which corresponds to a two-mode squeezed state of modes i and j, in the limit of infinite squeezing ($r \to \infty$). The input

state is then $\mathbf{\Gamma}^{in} = \lim_{r\to\infty} \bigoplus_i^N \boldsymbol{\sigma}_{i,i+1}(r)$, where we have set periodic boundary conditions so that $N+1 = 1$ in labeling the sites. The projection corresponds mathematically to taking a Schur complement (see Refs. [6, 8] for details), yielding an output pure Gaussian MPS of N modes on a ring with a CM

$$\mathbf{\Gamma}^{out} = \mathbf{\Gamma}_x - \mathbf{\Gamma}_{sx}^T (\mathbf{\Gamma}_{ss} + \boldsymbol{\theta}\mathbf{\Gamma}^{in}\boldsymbol{\theta})^{-1}\mathbf{\Gamma}_{sx}\,, \qquad (3)$$

where $\mathbf{\Gamma}_{sx} = \bigoplus_i \boldsymbol{\gamma}_{sx}$, and $\boldsymbol{\theta} = \bigoplus_i \mathrm{diag}\{1, 1, -1, -1\}$ represents transposition in phase space ($\hat{q}_i \to \hat{q}_i$, $\hat{p}_i \to -\hat{p}_i$).

First we note that the Gaussian states constructed in this way are ground states of harmonic Hamiltonians (a property of all Gaussian MPS [6]). This follows as no mutual correlations are created between the operators $\hat{q}_i$ and $\hat{p}_j$ for any $i, j = 1, \ldots, N$, having both EPR bonds and building blocks in standard form. The final CM Eq. (3) thus takes the form

$$\mathbf{\Gamma}^{out} = C^{-1} \oplus C\,, \qquad (4)$$

where C is a circulant $N \times N$ matrix. It can be shown that a CM of the form Eq. (4) corresponds to the ground state of the quadratic Hamiltonian $\hat{H} = \frac{1}{2}\left(\sum_i \hat{p}_i^2 + \sum_{i,j} \hat{q}_i V_{ij} \hat{q}_j\right)$, with the potential matrix given by $V = C^2$ [14].

Entanglement distribution.— In the Jamiolkowski picture [6, 8], different MPS projectors correspond to differently entangled Gaussian building blocks. Let us recall that, according to the "positivity of partial transposition" (PPT) criterion, a Gaussian state is separable (with respect to a $1 \times N$ bipartition) if and only if the partially transposed CM satisfies the uncertainty principle [15]. As a measure of entanglement, for two-mode symmetric Gaussian states $\boldsymbol{\gamma}_{i,j}$ the entanglement of formation E_F is computable via the formula [16]: $E_F(\boldsymbol{\gamma}_{i,j}) = \max\{0, f(\eta_{i,j})\}$, with $f(x) = \frac{(1+x)^2}{4x}\log\frac{(1+x)^2}{4x} - \frac{(1-x)^2}{4x}\log\frac{(1-x)^2}{4x}$. Here the positive parameter $\eta_{i,j}$ is the smallest symplectic eigenvalue of the partial transpose of $\boldsymbol{\gamma}_{i,j}$. For a two-mode state, $\eta_{i,j}$ can be computed from the symplectic invariants of the state [17], and the PPT criterion simply yields $\boldsymbol{\gamma}_{i,j}$ entangled as soon as $\eta_{i,j} < 1$, while infinite entanglement is reached for $\eta_{i,j} \to 0^+$.

We are interested in studying the quantum correlations of Gaussian MPS of the form as in Eq. (3), and in relating them to the entanglement properties of the building block $\boldsymbol{\gamma}$. The CM in Eq. (1) describes a physical state if $x \geq 1$ and $s \geq s_{\min} \equiv (x+1)/2$ [13]. At fixed x, and so at fixed CM of mode 3 (output port), the entanglement in the CM $\boldsymbol{\gamma}_{ss}$ of the first two modes (input port) is monotonically increasing as a function of s (as it can be checked by studying the respective symplectic eigenvalue η_{ss}), ranging from the case $s = s_{\min}$ when $\boldsymbol{\gamma}_{ss}$ is separable to the limit $s \to \infty$ when the block $\boldsymbol{\gamma}_{ss}$ is infinitely entangled. Accordingly, the entanglement between each of the first two modes of $\boldsymbol{\gamma}$ and the third one decreases with s. The main question we raise is how the initial entanglement in the building block $\boldsymbol{\gamma}$ gets distributed in the Gaussian MPS $\mathbf{\Gamma}^{out}$. The answer will be that the more entanglement we prepare in the input port $\boldsymbol{\gamma}_{ss}$, the longer the range of the quantum correlations in the output MPS will be. We start from the case of minimum s.
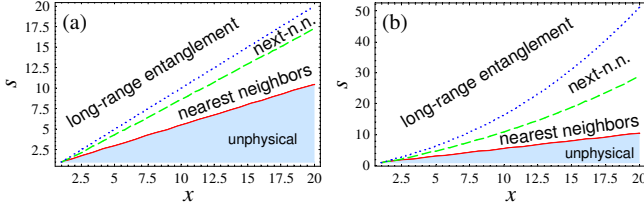
FIG. 2: (Color online) Entanglement distribution for a six-mode Gaussian MPS constructed from (a) infinitely entangled EPR bonds and (b) finitely entangled bonds given by two-mode squeezed states of the form Eq. (2) with $r = 1.1$. The entanglement thresholds s_k with $k = 1$ (solid red line), $k = 2$ (dashed green line) and $k = 3$ (dotted blue line) are depicted as functions of the parameter x of the building block. For $s > s_k$, all pairs of sites i and j with $|i - j| \leq k$ are entangled (see text for further details).
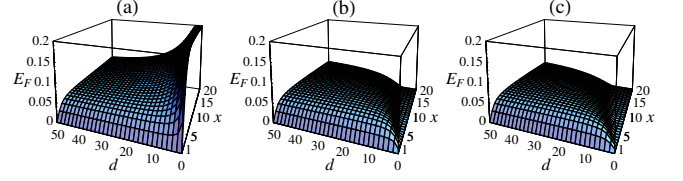

FIG. 3: (Color online) Entanglement of formation between two sites i and j in a six-mode Gaussian MPS, with $|i - j|$ equal to: (a) 1, (b) 2, and (c) 3, as a function of the parameters x and $d = s - s_{\min}$.

Short-range correlations.— Let us consider a building block γ with $s = s_{\min} = (x + 1)/2$. It is straightforward to evaluate, as a function of x, the Gaussian MPS in Eq. (3) for an arbitrary number of modes (we omit the CM here, as no particular insight can be drawn from the the explicit expressions of the covariances). By repeatedly applying the PPT criterion, one can analytically check that each reduced two-mode block $\gamma_{i,j}^{out}$ is separable for $|i - j| > 1$, which means that the output MPS Γ^{out} exhibits bipartite entanglement only between nearest neighbor modes, for any value of $x > 1$ (for $x = 1$ we obtain a product state). While this certainly entails that Γ^{out} is genuinely multiparty entangled, due to the translational invariance, it is interesting to observe that, without feeding entanglement in the input port γ_{ss} of the original building block, the range of quantum correlations in the output MPS is minimum. The pairwise entanglement between nearest neighbors will naturally decrease with increasing number of modes, being frustrated by the overall symmetry and by the intrinsic limitations on entanglement sharing (the so-called *monogamy* constraints [18]). We can study the asymptotic scaling of this entanglement in the limit $x \to \infty$. One finds that the corresponding symplectic eigenvalue $\eta_{i,i+1}$ is equal to $(N - 2)/N$ for even N, and $[(N - 2)/N]^{1/2}$ for odd N: neighboring sites are thus considerably more entangled if the ring size is even-numbered. Such frustration effect on entanglement in odd-sized rings, already devised in a similar context in Ref. [19], is quite puzzling. An explanation may follow from counting arguments applied to the number of parameters (which are related to the degree of pairwise entanglement) characterizing a generic pure state on harmonic lattices [20].

Medium-range correlations.— The connection between input entanglement and output correlation length can be investigated in detail considering a general γ with $s > s_{\min}$. The MPS CM in Eq. (3) can still be worked out analytically for a low number of modes, and numerically for higher N. Let us keep the parameter x fixed; we find that with increasing s the correlations extend smoothly to distant modes. A series of thresholds s_k can be found such that for $s > s_k$, two given modes i and j with $|i - j| \leq k$ are entangled. While trivially

$s_1(x) = s_{\min}$ for any N (notice that nearest neighbors are entangled also for $s = s_1$), the entanglement boundaries for $k > 1$ are in general different functions of x, depending on the number of modes. We observe however a certain regularity in the process: $s_k(x, N)$ always increases with the integer k. These considerations follow from analytic calculations on up to ten-modes MPS, and we can infer them to hold true for higher N as well, given the overall scaling structure of the MPS construction process. Very remarkably, this means that the maximum range of bipartite entanglement between two modes, i.e. the maximum distribution of multipartite entanglement, in a Gaussian MPS on a translationally invariant ring, is *monotonically* related to the amount of entanglement in the reduced two-mode input port of the building block.

To clearly demonstrate this intriguing connection, let us consider the example of a Gaussian MPS with $N = 6$ modes. In a six-site translationally invariant ring, each mode can be correlated with another being at most 3 sites away ($k = 1, 2, 3$). From a generic building block Eq. (1), the 12×12 CM Eq. (3) can be analytically computed as a function of s and x. We can construct the reduced CMs $\gamma_{i,i+k}^{out}$ of two modes with distance k, and evaluate for each k the respective symplectic eigenvalue $\eta_{i,i+k}$ of the corresponding partial transpose. The entanglement condition $s > s_k$ will correspond to the inequality $\eta_{i,i+k} < 1$. With this conditions one finds that $s_2(x)$ is the only acceptable solution to the equation: $72s^8 - 12(x^2 + 1)s^6 + (-34x^4 + 28x^2 - 34)s^4 + (x^6 - 5x^4 - 5x^2 + 1)s^2 + (x^2 - 1)^2(x^4 - 6x^2 + 1) = 0$, while for the next-next-nearest neighbors threshold one has simply $s_3(x) = x$. This enables us to classify the entanglement distribution and, more specifically, to observe the interaction scale in the MPS Γ^{out}: Fig. 2(a) clearly shows how, by increasing initial entanglement in γ_{ss}, one can gradually switch on quantum correlations between more and more distant sites.

We can also study entanglement quantitatively. Fig. 3 shows the entanglement of formation E_F of $\gamma_{i,i+k}^{out}$ for $k = 1, 2, 3$ (being computable in such symmetric two-mode reductions), as a function of x and $d \equiv s - s_{\min}$. For any (x, d) the entanglement is a decreasing function of the integer k, i.e. quite naturally it is always stronger for closer sites. However, in the limit of high d (or, equivalently, high s), the three surfaces become close to each other. We want now to deal exactly with this limit, for a generic number of modes.

Long-range correlations.— In the limit $s \to \infty$, the expressions greatly simplify and we obtain a N-mode Gaussian

MPS $\mathbf{\Gamma}^{out}$ of the form Eq. (4), where C and C^{-1} are completely degenerate circulant matrices, with $(C^{-1})_{i,i} = a_q = [(N-1)+x^2]/(Nx)$, $(C^{-1})_{i,j\neq i} = c_q = (x^2-1)/(Nx)$; and accordingly $(C)_{i,i} = a_p = [1+(N-1)x^2]/(Nx)$, $(C^{-1})_{i,j\neq i} = c_p = -c_q$. For any N, thus, each individual mode is *equally entangled* with any other, no matter how distant they are.

The asymptotic limit of our analysis shows then that an infinitely entangled input port of the building block results in a MPS with *maximum* pairwise entanglement length. These N-mode Gaussian states are well-known as useful resources for multiparty CV communication protocols [12]. The CM $\mathbf{\Gamma}^{out}$ of these MPS can in fact be put, by local symplectic (unitary) operations, in a standard form parametrized by the single-mode purity $\mu_{loc} = (a_q a_p)^{-1/2}$ [9]. Remarkably, in the limit $\mu_{loc} \to 0$ (i.e. $x \to \infty$), the entropy of any K-sized ($K < N$) sub-block of the ring, quantifying entanglement between K modes and the remaining $N-K$, is *infinite* [9]. Within the MPS framework, we also understand the peculiar "promiscuous" entanglement sharing [18] of these fully symmetric states: being them built by a symmetric distribution of infinite pairwise entanglement among multiple modes, they achieve maximum genuine multiparty entanglement while keeping the strongest possible bipartite one in any pair. Let us note that in the field limit ($N \to \infty$) each single pair of modes is in a separable state, as they have to mediate a genuine multipartite entanglement distributed among *all* the infinite modes [9].

Finitely entangled bonds and experimental feasibility.— We finally discuss how to implement the recipe of Fig. 1 to produce Gaussian MPS experimentally. The three-mode building blocks can be engineered for any choice of (x,s) (within the practical limitation of a finite available degree of squeezing) by combining three single-mode squeezed states through a sequence of up to three beam splitters [13]. The EPR measurements are in turn realized by homodyne detections [8].

The only unfeasible part of the scheme is constituted by the ancillary EPR pairs. But are *infinitely* entangled bonds truly necessary? One could consider a $\mathbf{\Gamma}^{in}$ given by the direct sum of two-mode squeezed states of Eq. (2), but with finite r. Repeating our analysis to investigate the entanglement properties of the resulting Gaussian MPS with finitely entangled bonds, we find that, at fixed (x,s), the entanglement in the various partitions is degraded as r decreases, as somehow expected. Crucially, this does not affect the connection between input entanglement and output correlation length. Numerical investigations show that, while the thresholds s_k for the onset of entanglement between distant pairs are quantitatively modified – a bigger s is required at a given x to compensate the less entangled bonds – the overall structure stays untouched. As an example, Fig. 2(b) depicts the entanglement distribution in six-mode MPS obtained from finitely entangled

bonds with $r = 1.1$, corresponding to ≈ 6.6 dB of squeezing. Single-bonded Gaussian MPS, which surprisingly encompass a broad class of physically relevant multimode states, can thus be experimentally produced, and their entanglement distribution can be precisely engineered starting from the parameters of a simple bisymmetric three-mode building block, with the supply of two-mode *finitely* squeezed states.

Concluding remarks.— We have shown that the range of pairwise quantum correlations in translationally invariant N-mode Gaussian MPS is determined by the entanglement in the input port of the building block. As a consequence of this interesting connection, a striking difference between finite-dimensional and infinite-dimensional MPS is unveiled, as the former are by construction slightly entangled for a low dimensionality of the bonds [10], and their entanglement is short-ranged [11]. We proved instead that pure, fully symmetric, N-mode Gaussian states are exactly MPS with minimum bond cardinality ($M = 1$): yet, their entanglement can *diverge* across any global splitting of the modes, and their pairwise quantum correlations have *maximum* range. How this feature connects with the potential validity of an area law for critical bosonic systems is currently an open question.

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